Ph.D. Course on
"Vorticity, Vortical Flows and Vortex-Induced Vibrations"
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Lecture 3

Vortex filaments

Thomas Leweke

IRPHE, CNRS, Marseille, France

Institut de Recherche sur les Phénomènes Hors Equilibre

Examples of vortex filaments

Nomenclature and definitions

- **velocity**

$$\vec{u} = [u(x,y,z,t),\ v(x,y,z,t),\ w(x,y,z,t)]$$

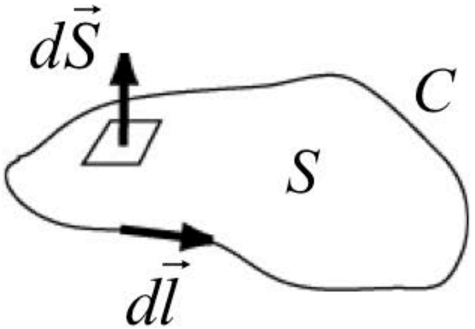
- **vorticity**

$$\vec{\omega} = \vec{\nabla} \times \vec{u} = \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y},\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z},\ \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right)$$

$$\vec{\nabla} = \left(\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \right) \qquad \Rightarrow \quad \vec{\nabla} \cdot \vec{\omega} = 0$$

- **circulation**

$$\Gamma = \oint_C \vec{u} \cdot d\vec{l} = \int_S \vec{\omega} \cdot d\vec{S}$$



Common hypotheses

- **Newtonian fluid**
  - ↳ **stresses $\propto$ velocity gradients**

- **constant-density fluid, $\rho\,(x,y,z,t)$ = const.**
  - ↳ **incompressible**
  - ↳ **barotropic**

- **conservative volume forces**
  - ↳ $\vec{F} = -\vec{\nabla}\Phi$
  - ↳ **example: gravity**

Balance and evolution equations

- Conservation of mass ("continuity")

$$\vec{\nabla}\vec{u} = 0$$

- Navier-Stokes equation

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p' + \nu\,\Delta\vec{u}$$

- Vorticity equation

$$\frac{D\vec{\omega}}{Dt} = (\vec{\omega}\cdot\vec{\nabla})\vec{u} + \nu\,\Delta\vec{\omega}$$

$\Delta = \nabla^2$
Laplacian

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + (\vec{u}\cdot\vec{\nabla})$$

material derivative

$p' = p + \rho\,\Phi$
p: pressure

ν: kinematic viscosity

Laws and theorems

➢ **Biot-Savart relation**

$$\vec{u}(\vec{r},t) = -\frac{1}{4\pi}\int_V \frac{(\vec{r}-\vec{r}\,')\times\vec{\omega}(\vec{r}\,',t)}{\left|\vec{r}-\vec{r}\,'\right|^3}\,d^3r'$$

➢ **Kelvin's Theorem for an ideal fluid ($\nu = 0$)**

"The circulation of any closed material line is conserved during its motion"

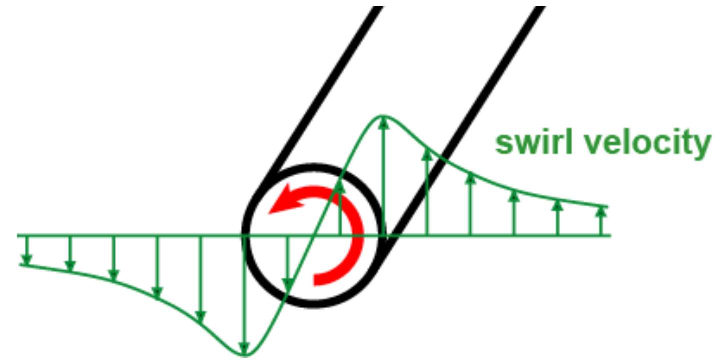
$$\frac{D\Gamma}{Dt} = 0$$

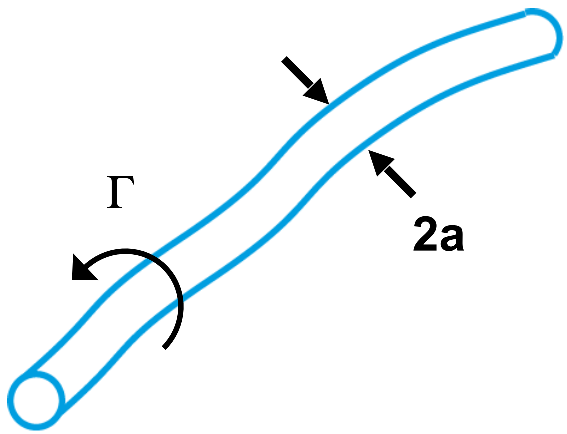
↳ **Theorems and laws of Lagrange and Helmholtz**

↳ ***summary:*** **In an ideal fluid, the circulation of each fluid element is constant in time and advected by the velocity field**

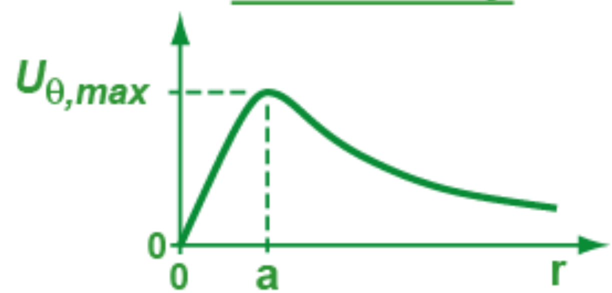
Vortices

- local concentration of vorticity
- (fairly) axisymmetric
- tube-like structure

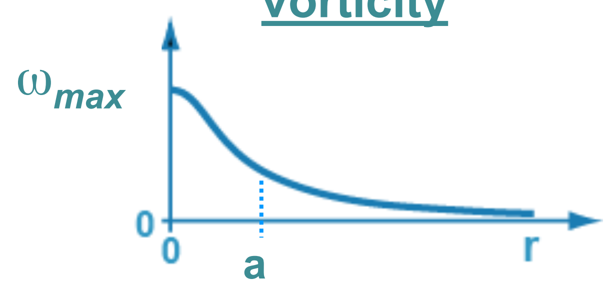




swirl velocity

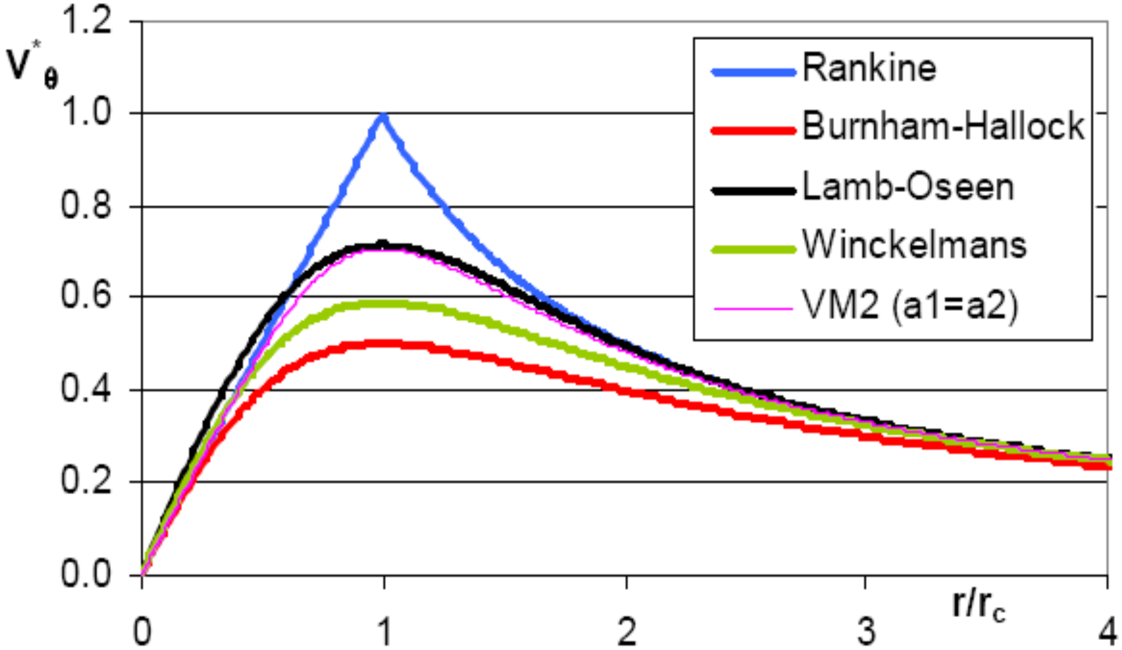


vorticity



- circulation Γ
- core radius a
- Reynolds number $Re = \Gamma / \nu$

Velocity profiles for various vortex models



Velocity profiles, different analytical models.

Rankine vortex

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r_c}\frac{r}{r_c} \quad \text{for } r \leqslant r_c,$$

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r} \quad \text{for } r > r_c.$$

Lamb–Oseen vortex [75]

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r}\{1 - \exp(-1.2526(r/r_c)^2)\}.$$

Hallock–Burnham vortex [7]

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r}\frac{r^2}{r^2 + r_c^2}.$$

Smooth blending vortex profile (Winckelmans et al. [124])

$$v_\theta(r) = \frac{\Gamma_0}{2\pi r}\left\{1 - \exp\left(-\frac{\beta_i(r/B)^2}{\{1 + [(\beta_i/\beta_o)(r/B)^{5/4}]^p\}^{1/p}}\right)\right\},$$

with β_o, β_i, and $p = 10$, 500, and 3, respectively.

Multiple scale vortex (Jacquin et al. [65])

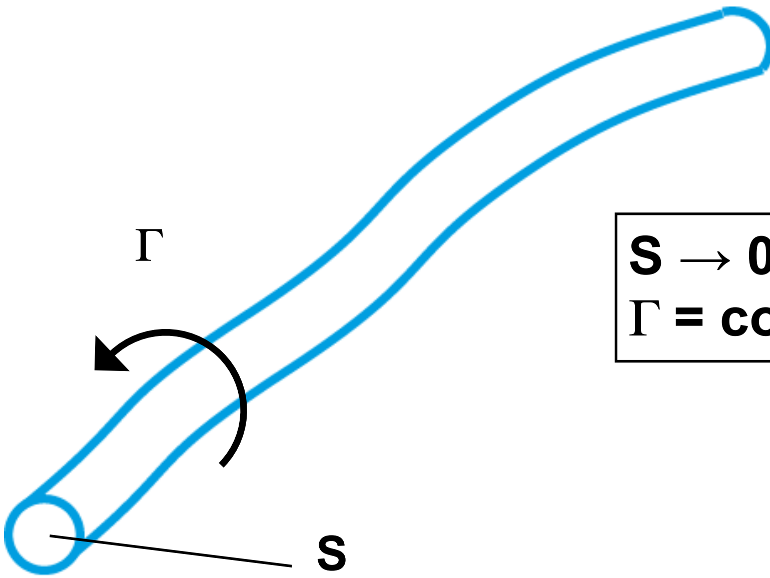
$$v_\theta(r) = \frac{\Gamma_0}{2\pi r_i}\frac{r}{(r_i r_o)^{1/2}} \quad \text{for } r \leqslant r_i$$

$$v_\theta(r) = \frac{\Gamma_0}{2\pi(r_o r)^{1/2}} \quad \text{for } r_i \leqslant r \leqslant r_o$$

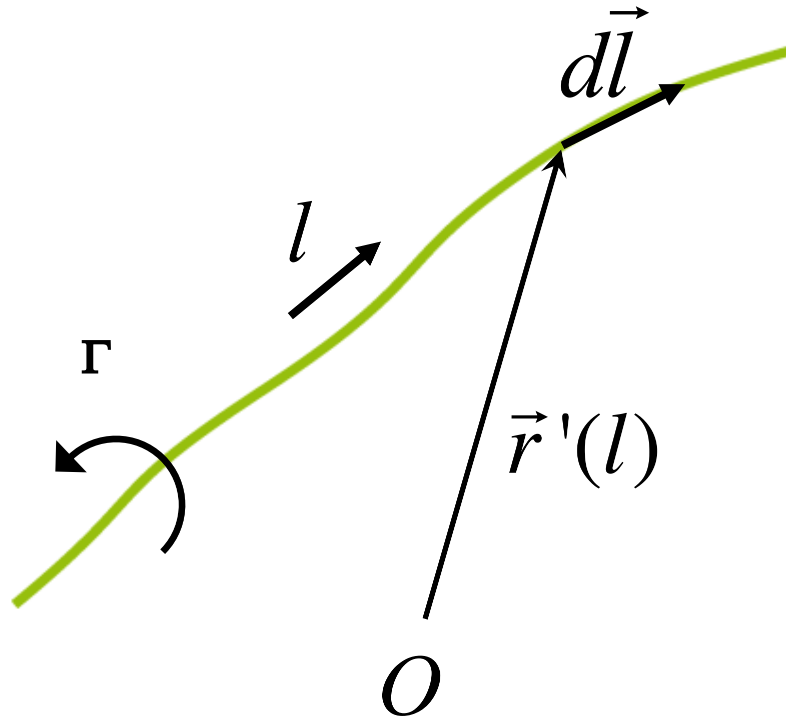
$$v_\theta(r) = \frac{\Gamma_0}{2\pi r} \quad \text{for } r \geqslant r_o$$

Gerz et al. (2002)

Vortex filaments



$S \rightarrow 0$
Γ = const.



Evolution of filament shape:

- calculate $\vec{u}[\vec{r}(l)] = \vec{u}_{ext} + \vec{u}_{ind}$

- using Biot-Savart

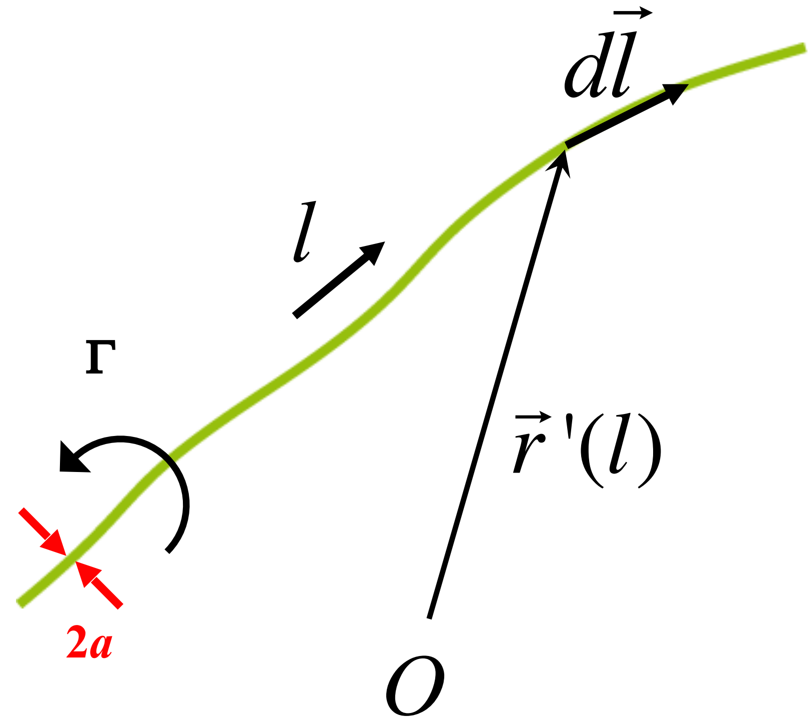
$$\vec{\omega} d^3 r' \rightarrow \Gamma d\vec{l}$$

$$\vec{u}_{ind}(\vec{r},t) = -\frac{\Gamma}{4\pi}\int_L \frac{(\vec{r}-\vec{r}\,')\times d\vec{l}}{\left|\vec{r}-\vec{r}\,'\right|^3}$$

Vortex filaments

Problem:

- singularity for $\vec{r} = \vec{r}\,'$



Solution:

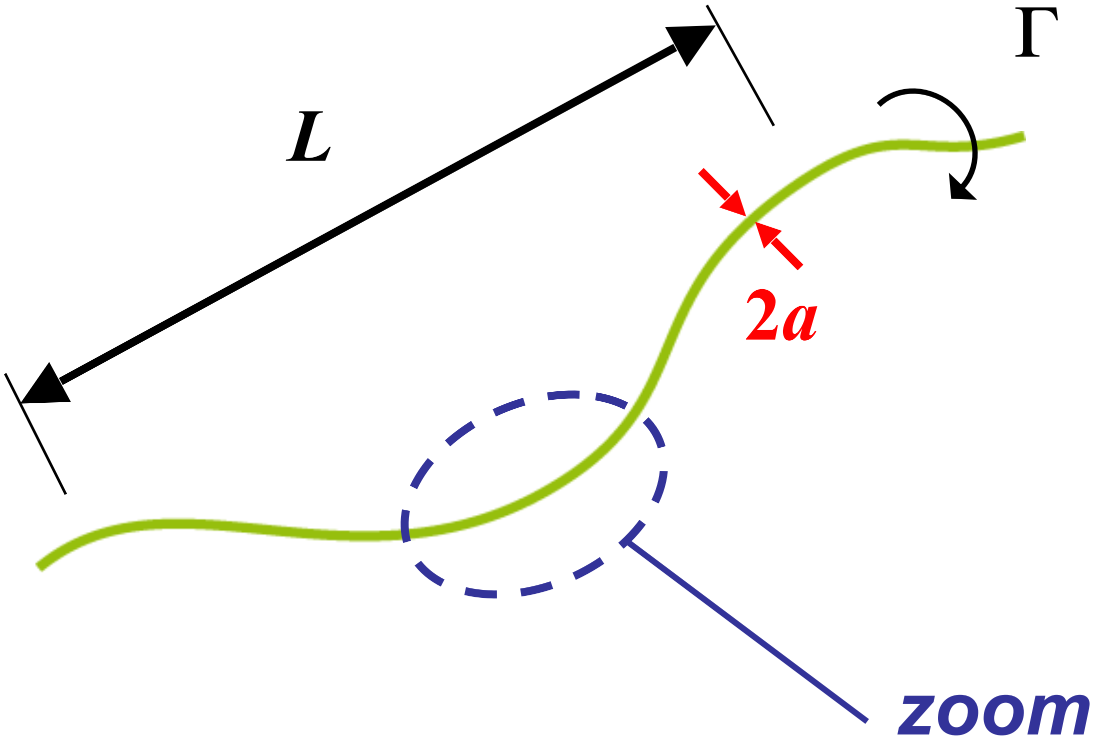
- reconsider finite core size a

- $a \ll R_o$, $a \ll L$

$$\vec{u}_{ind}(\vec{r},t) = -\frac{\Gamma}{4\pi}\int_L \frac{(\vec{r}-\vec{r}\,')\times d\vec{l}}{\left|\vec{r}-\vec{r}\,'\right|^3}$$

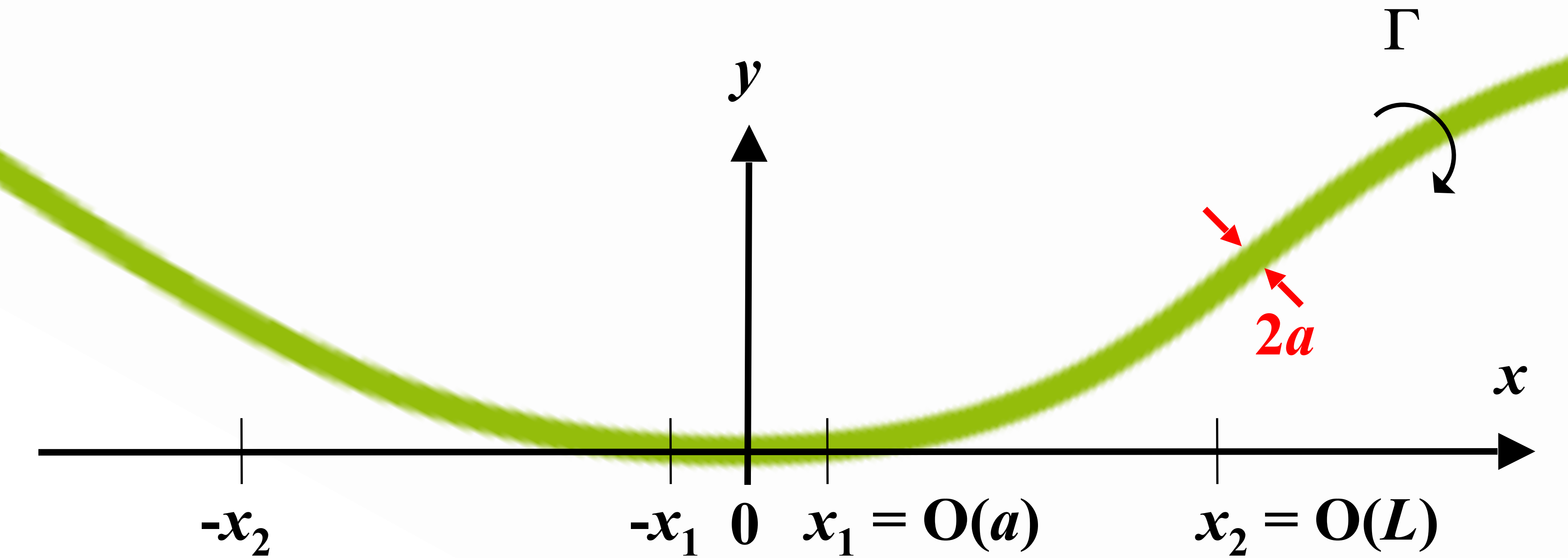
Vortex filament evolution

Local Induction Approximation



Vortex filament evolution

Local Induction Approximation

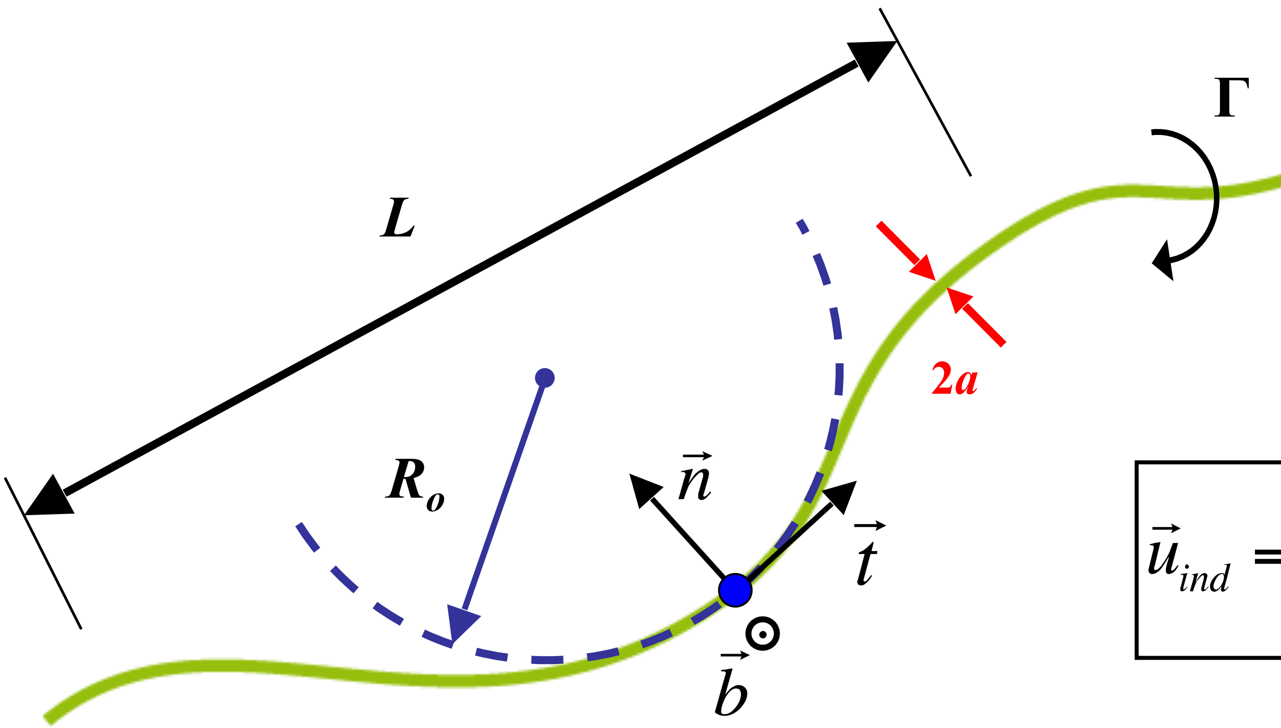


$|x| < x_1$: **no filament**

$|x| < x_2$: **center line** $y \sim x^2$

Vortex filament evolution

Local Induction Approximation

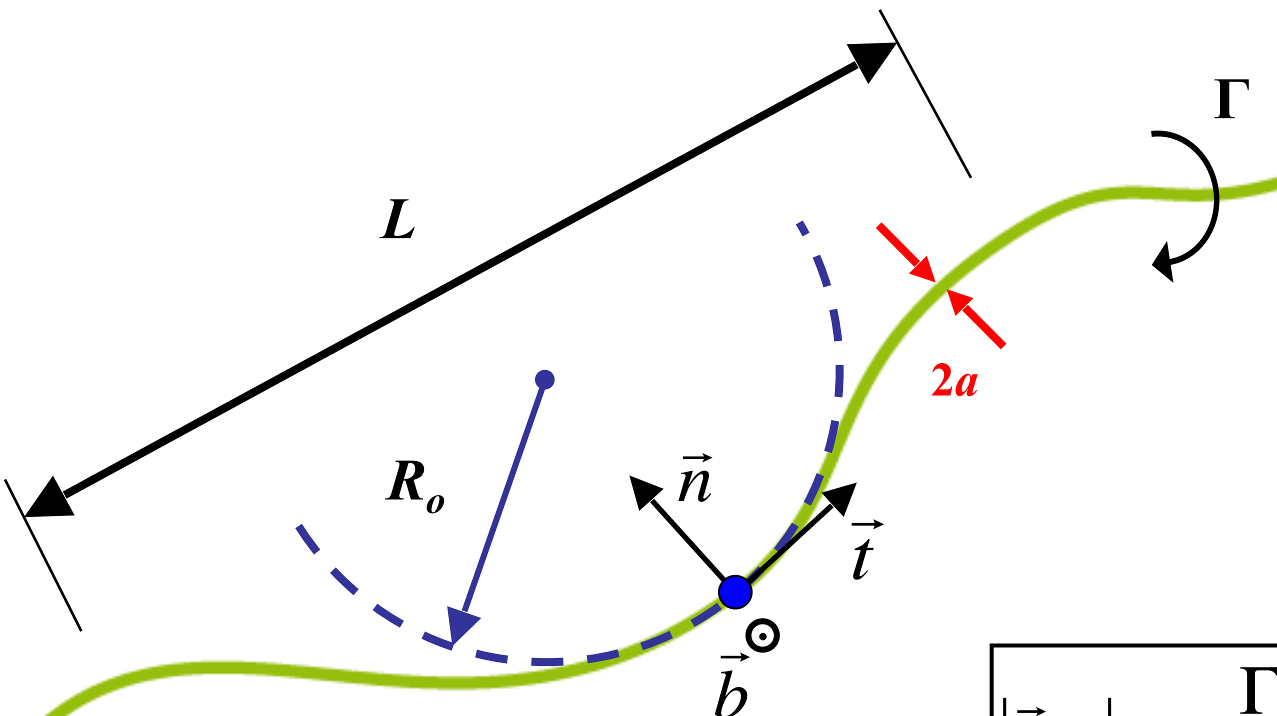


$$\vec{u}_{ind} = \frac{\Gamma}{4\pi R_o} \ln \frac{L}{a} \; \vec{b}$$

error of $O\left(\frac{\Gamma}{4\pi R_o}\right)$

Vortex filament evolution

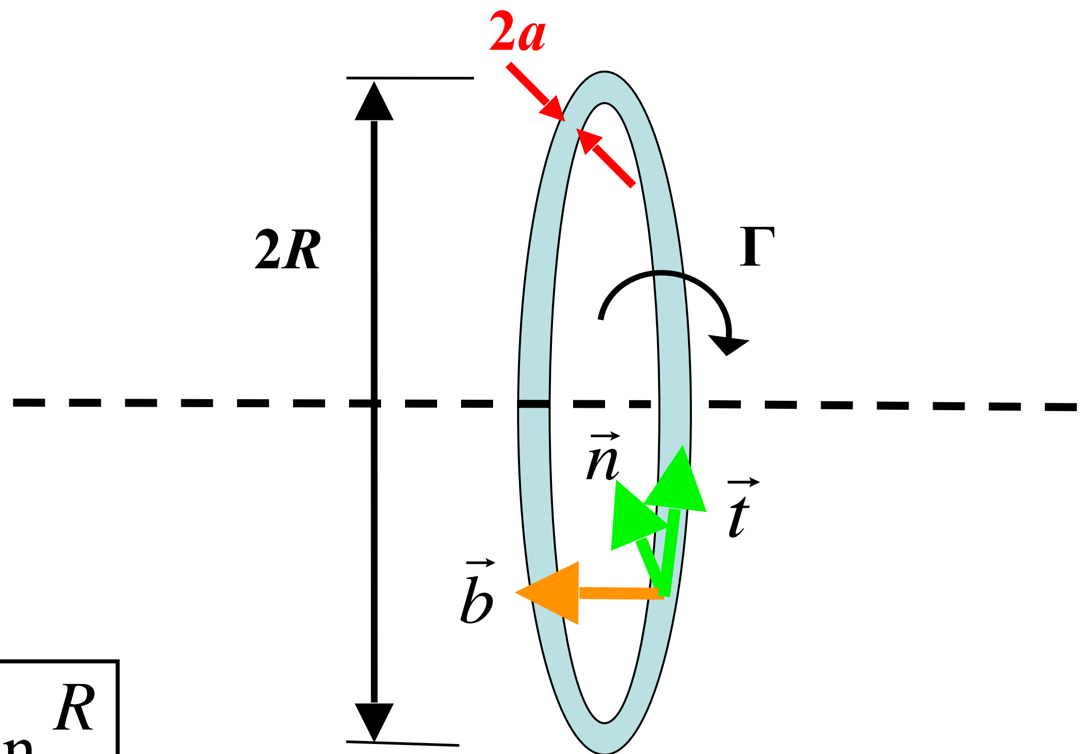
Local Induction Approximation



$$\left|\vec{u}_{ind}\right| = \frac{\Gamma}{4\pi R_o}[\ln\frac{L}{a} + O(1)]$$

Examples

Vortex ring



$L = R$

L.I.A.:

$$\left|\vec{u}_{ind}\right| = \frac{\Gamma}{4\pi R_o} \ln \frac{R}{a}$$

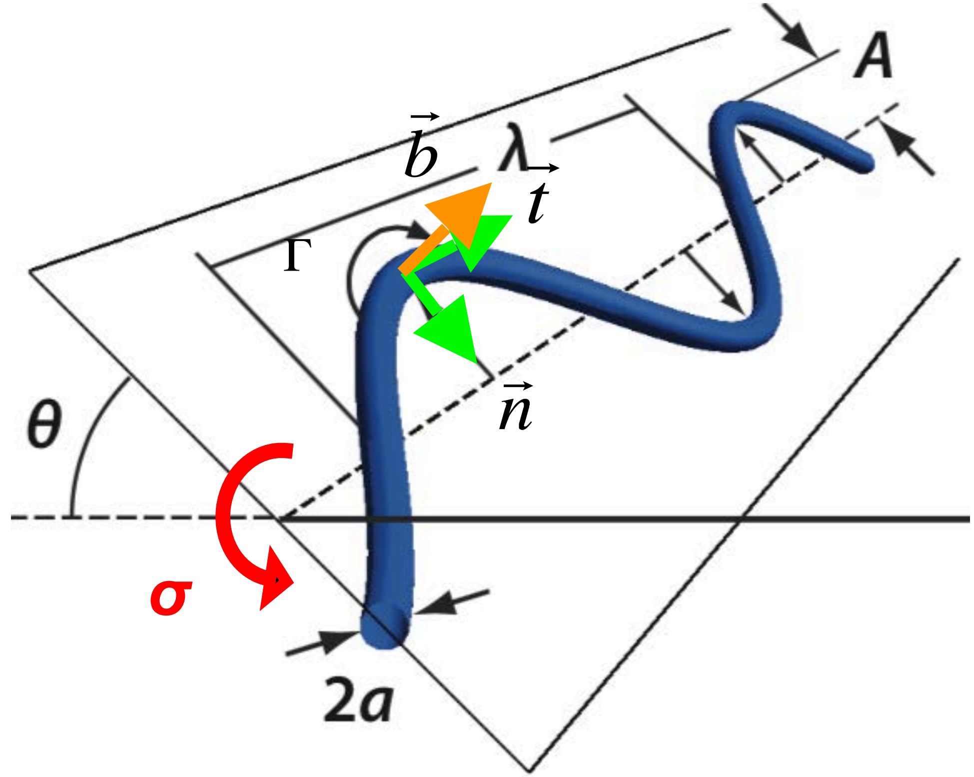
Exact (Kelvin 1867):

$$\left|\vec{u}_{ind}\right| = \frac{\Gamma}{4\pi R_o} [\ln \frac{8R}{a} - \frac{1}{4}]$$

+75% for $a/R = 0.1$

Examples

Sinusoidal filament



$$L = 1/k = \lambda/2\pi$$

L.I.A.:

$$\sigma = -\frac{\Gamma}{4\pi} k^2 \ln(ka)$$

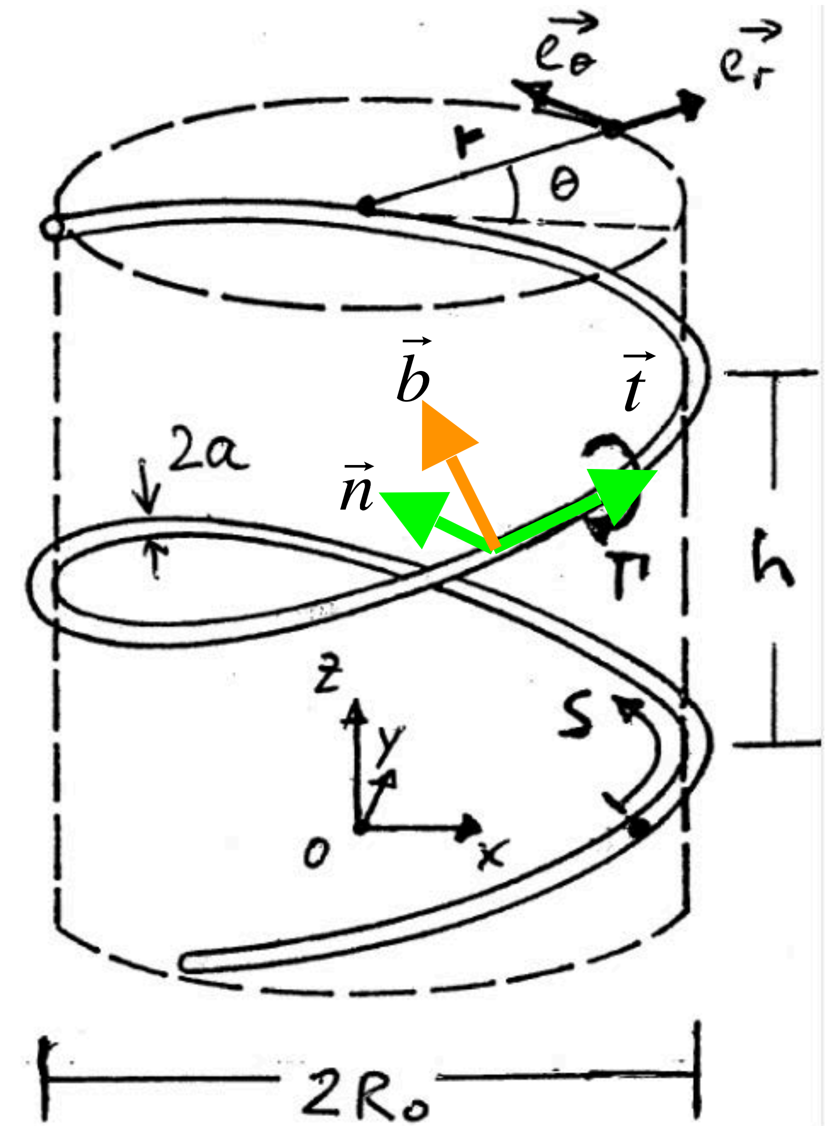
Exact:

$$\sigma = -\frac{\Gamma}{4\pi} k^2 [\ln(ka) - \ln 2 + C + \frac{1}{4}]$$

Examples

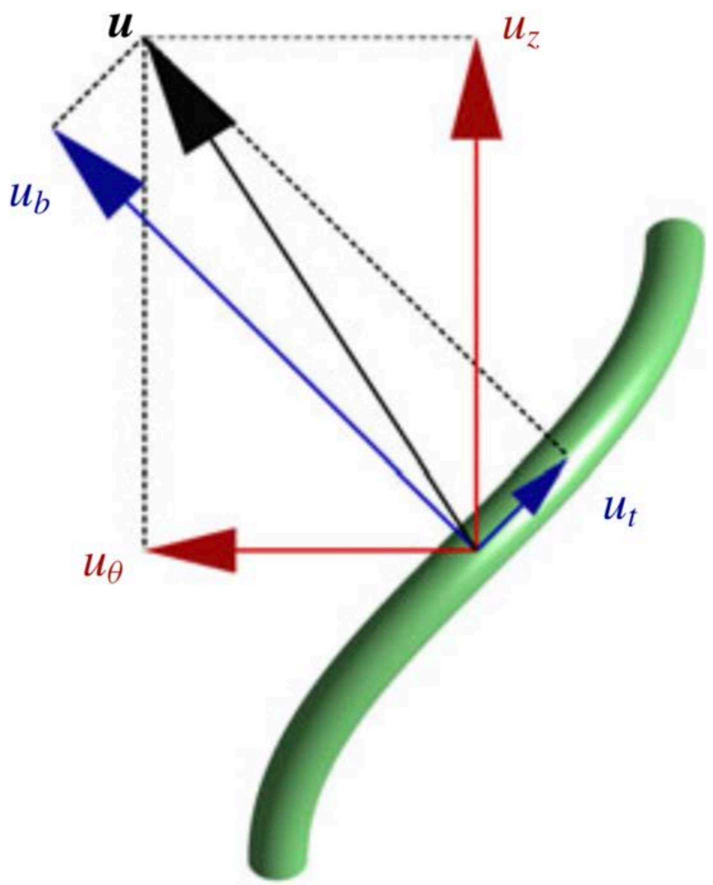
Helical vortex

Local-Induction Approximation



Examples

Helical vortex



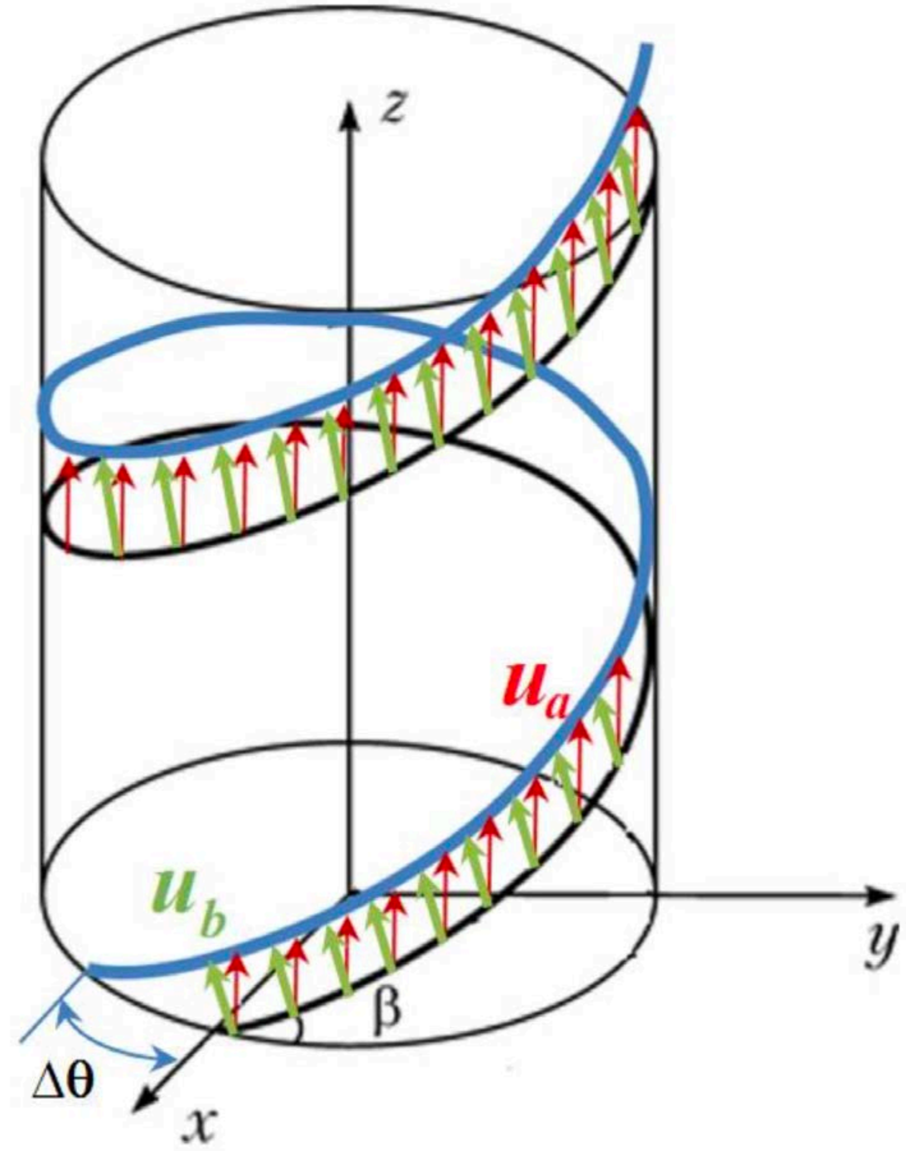
Exact calculation:

Non-zero tangential velocity

Examples

Helical vortex

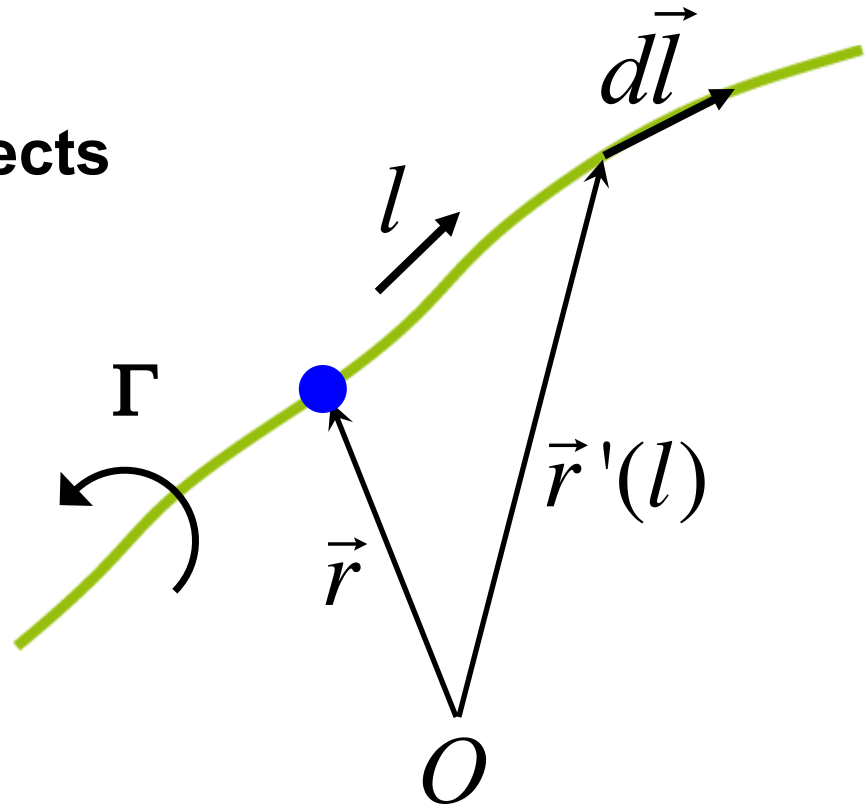
"Motion" of the helical filament



Vortex filament evolution

Cut-off method

- include non-local effects

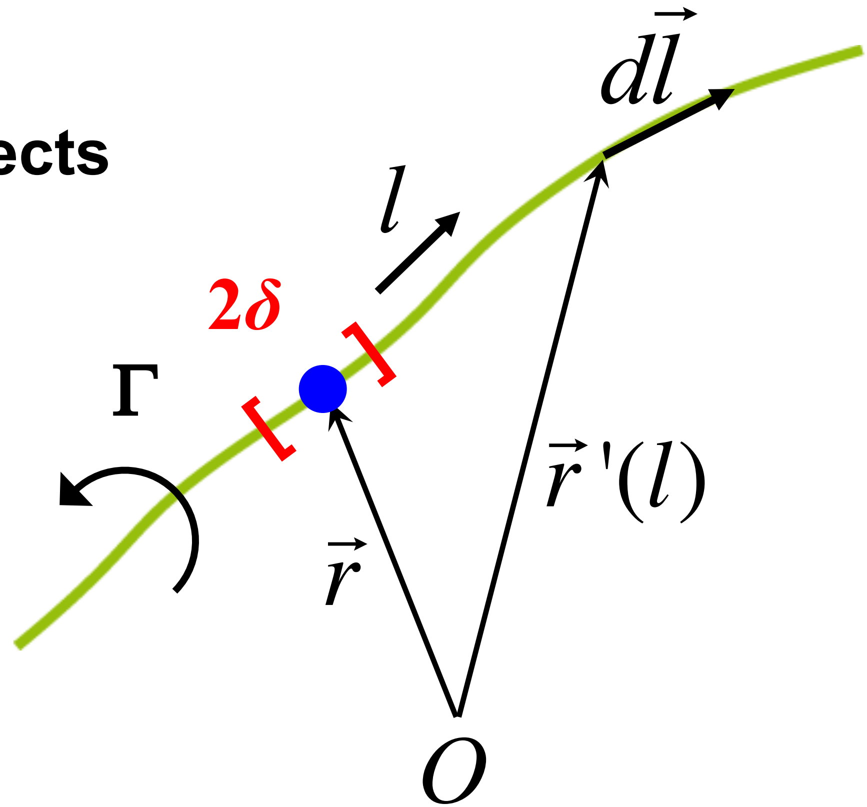


$$\vec{u}_{ind}(\vec{r},t) = -\frac{\Gamma}{4\pi}\int_L \frac{(\vec{r}-\vec{r}\,')\times d\vec{l}}{|\vec{r}-\vec{r}\,'|^3}$$

Vortex filament evolution

Cut-off method

- include non-local effects



Which δ?

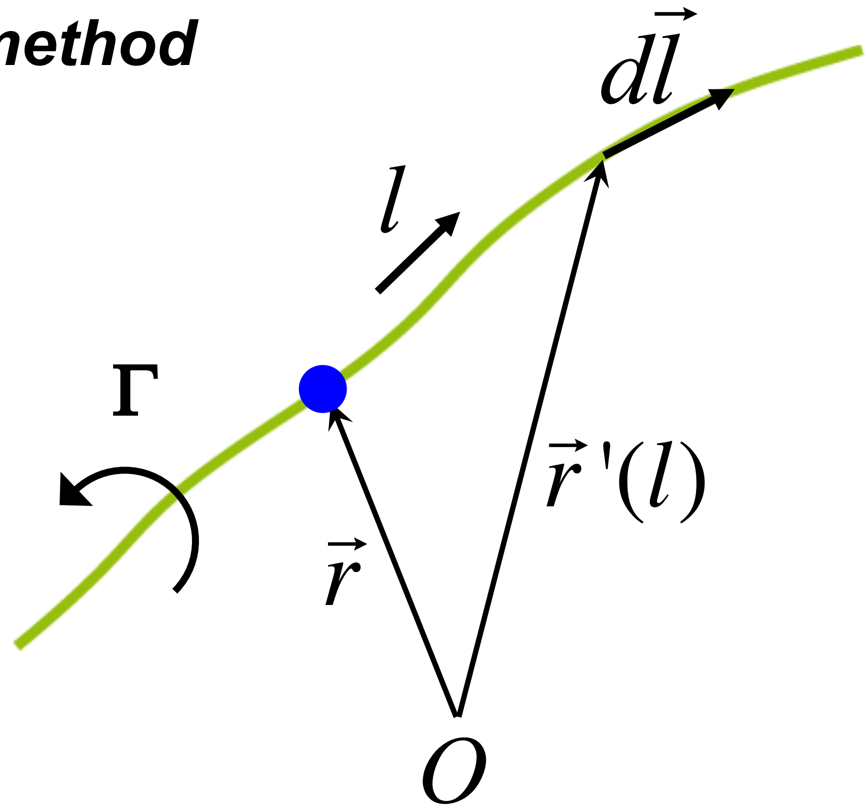
$$\delta = (a/2)\exp(1/4)$$

for a Rankine vortex

$$\vec{u}_{ind}(\vec{r},t) = -\frac{\Gamma}{4\pi}\int_{[\delta]} \frac{(\vec{r}-\vec{r}\,')\times d\vec{l}}{|\vec{r}-\vec{r}\,'|^3}$$

Vortex filament evolution

Alternative to cut-off method



Which μ?

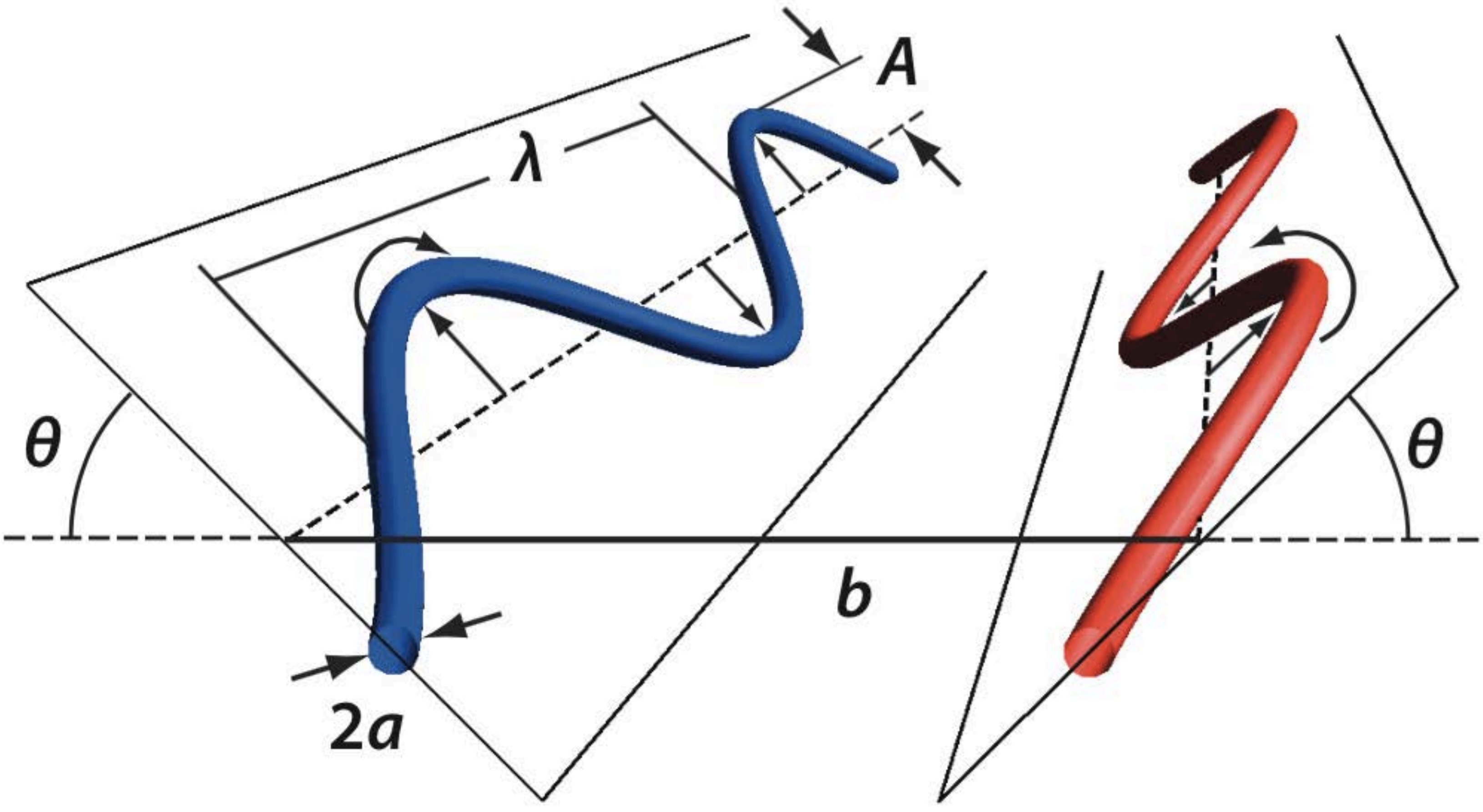
$$\mu = a \exp(-3/4)$$

for a Rankine vortex

$$\vec{u}_{ind}(\vec{r},t) = -\frac{\Gamma}{4\pi}\int_L \frac{(\vec{r}-\vec{r}\,')\times \vec{dl}}{\left[(\vec{r}-\vec{r}\,')^2+\mu^2\right]^{3/2}}$$

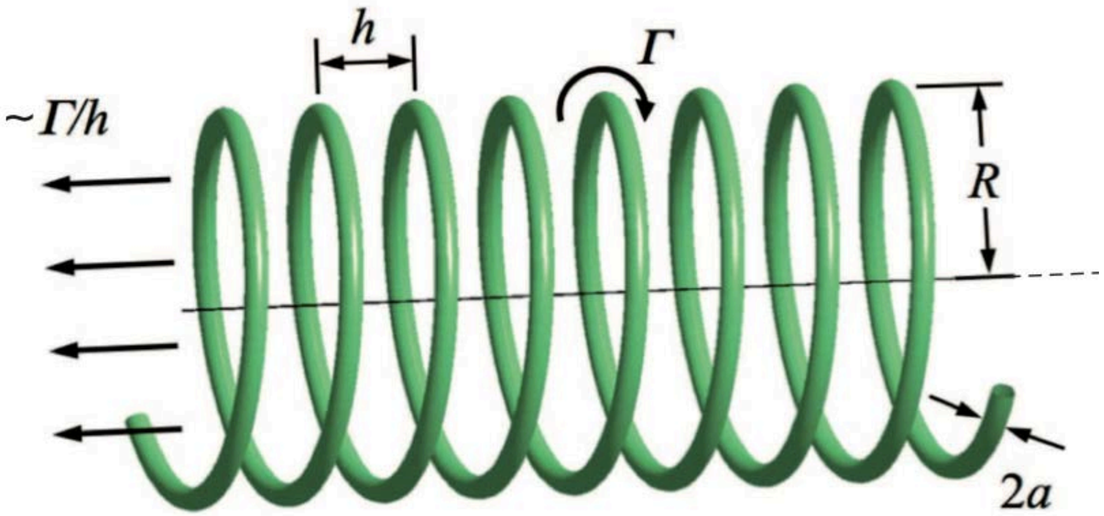
Examples

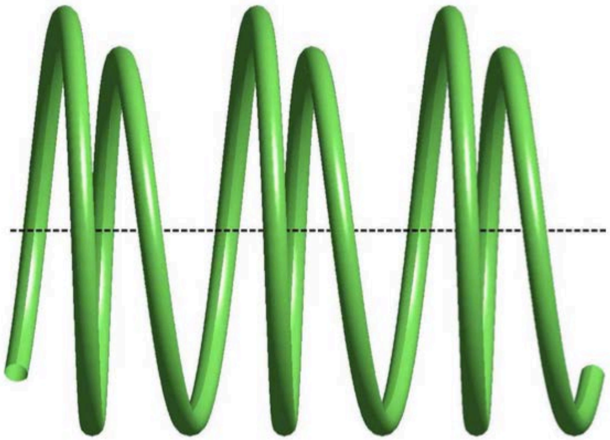
Crow instability of a counter-rotating vortex pair **(Crow 1970)**

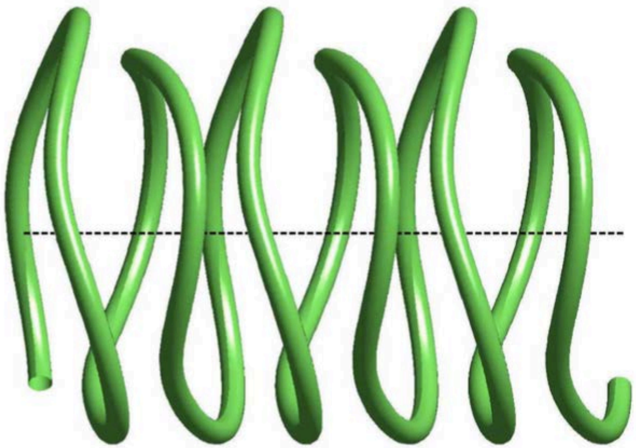


Examples

Pairing instability of a helical vortex







Takeaways

- ➡ Vortices (vortex filaments) are transported with the flow
- ➡ Vortices ‘generate’ a flow
- ➡ Curved filaments possess a self-induced velocity
- ➡ Description of evolution by desingularized 1-D Biot-Savart relation
  - Local-Induction Approximation:
    motion in bi-normal direction ($\perp$ to plane of curvature)
  - In general:
    motion in tangential direction possible

Literature

Crow, S. C. 1970 Stability theory for a pair of trailing vortices.
AIAA Journal **8**, 2172-2179

Gerz T., Holzäpfel F., Darracq, D. 2002 Commercial aircraft wake vortices.
Progress in Aerospace Sciences **38**, 181-208

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Wu, J.-Z., Ma, H.-Y., Zhou, M.-D. 2006 *Vorticity and vorticity dynamics*.
Springer. Chapters 3.2 and 8.2.

End of Lecture 3